Theoretical and Experimental Study of Locally Resonant and Bragg Band Gaps in Flexural Beams Carrying Periodic Arrays of Beam-Like Resonators

In this paper, we present a design of locally resonant (LR) beams using periodic arrays of beam-like resonators (or beam-like vibration absorbers) attached to a thin homogeneous beam. The main purpose of this work is twofold: (i) providing a theoretical characterization of the proposed LR beams, including the band gap behavior of infinite systems and the vibration transmittance of finite structures, and (ii) providing experimental evidence of the associated band gap properties, especially the coexistence of LR and Bragg band gaps, and their evolution with tuned local resonance. For the first purpose, an analytical method based on the spectral element formulations is presented, and then an in-depth numerical study is performed to examine the band gap effects. In particular, explicit formulas are provided to enable an exact calculation of band gaps and an approximate prediction of band gap edges. For the second purpose, we fabricate several LR beam specimens by mounting 16 equally spaced resonators onto a free-free host beam. These specimens use the same host beam, but the resonance frequencies of the resonators on each beam are different. We further measure the vibration transmittances of these specimens, which give evidence of three interesting band gap phenomena: (i) transition between LR and Bragg band gaps; (ii) near-coupling effect of the local resonance and Bragg scattering; and (iii) resonance frequency of local resonators outside of the LR band gap. [DOI: 10.1115/1.4024214]

Keywords: periodic beam structure, band gap, local resonance, Bragg scattering, phononic crystal, dynamic vibration absorber

1 Introduction

Wave motion in periodic structures has been investigated for many years [1–4]. It is well known that periodic structures can act as mechanical filters for wave propagation, thus meaning waves cannot propagate freely within specific frequency ranges called “band gaps” or “stop bands.” Mead [4] has made very substantial contributions to the analysis and prediction of wave motion in periodic engineering structures. In 1996, Mead [4] presented a comprehensive review of this subject. A large variety of periodic structures have been considered. Typical examples include periodic spring-mass systems [5–7], periodic rods [8–11], multisupported beams [1,12], lattice structures [13–16], etc. In contrast with conventional periodic structures, artificial periodic composites, known as phononic crystals (PCs) or phononic materials, have attracted renewed attention recently, with more attention paid to new wave phenomena and wave physics [17,18].

The unique band gap characteristics of periodic structures and PCs show promising applications, such as sound barriers, vibration isolators, acoustic filters, etc. From a practical perspective of the noise and vibration control engineering, broadband and low-frequency band gaps are generally desired. To achieve broadband band gaps, a proven method is using topology optimization techniques to maximize band gap sizes [19–24]. While realizing low-frequency band gaps, an interesting idea was recently proposed by Liu et al. [18], who construct locally resonant (LR) PCs using arrays of resonant units embedded in a host material. This idea has attracted much interest in the past few years [25,26]. Emphasis is placed on the engineering of LR band gaps in a frequency range much lower than that given by the Bragg limit. The concept of LR PCs has been subsequently implemented in the context of engineering structures. By mounting periodic arrays of local resonators to structural waveguides, such as rods [27,28], beams [29–32], and plates [33–35], a variety of LR-type periodic structures were proposed, showing the existence of LR band gaps. Earlier attention on LR periodic structures is focused on the situation that the resonance frequency of local resonators is tuned well below the Bragg condition. As a result, an LR band gap is opened in a low-frequency range and no visible Bragg-type band gaps can be observed. For instance, Wang et al. [27] presented theoretical and experimental evidences of the existence of a low-frequency LR band gap for longitudinal waves in an LR rod, while Yu et al. [29] evidenced the opening of a low-frequency LR band gap for flexural waves in an LR beam through both numerical simulations and experimental measurements.

Recently, investigations on several types of LR periodic structures showed that, by appropriately tuning the resonance frequency of local resonators, ultrawide LR and Bragg band gaps can coexist in such systems [28,30,32,36]. Xiao et al. [28,36] explored the evolution of LR and Bragg band gaps in LR strings/rods with tuning local resonance and found that, when the resonance frequency of the local resonators is tuned precisely to the Bragg condition, a superwide coupled LR–Bragg band gap can be...
form. Liu and Hussein [30] examined the evolution of frequency band structure of an LR beam as the spring constant or mass of local resonators is varied. In particular, they observed the location, width, and wave attenuation performance of both LR and Bragg band gaps [32]. They further derived analytical band edge frequency equations to explain the evolution of band gap behavior. In addition, they also provided explicit formulas, enabling direct predictions of the conditions for two special band gap states: (i) band gap near-coupling state that gives rise to a super-wide pseudogap, within which only a very narrow pass band exists, and (ii) band gap transition state that has been addressed by Liu and Hussein in Ref. [30]. It should be noted that the results of Refs. [28], [30], [32], and [36] were all based on theoretical predictions. Up to now, little work has given experimental evidence of the coexistence and evolution of LR and Bragg band gaps in LR periodic structures.

In this work, we present a design of LR beams where the local resonators are simply realized by double-ended thin beams [37–39] mounted on a thin homogeneous beam, as shown in Fig. 1(a). The double-ended thin beams are referred to as beam-like resonators in this paper. Taking notice of existing publications on LR beams [29–32], the main contribution of the present work is intended to be two-fold. On the one hand, we present an alternative design of the LR beam that can be realized easily and provide a systematic characterization of the designed LR beam structures accompanied with explicit design formulas. On the other hand, we provide experimental evidence of the coexistence and evolution of LR and Bragg band gaps in the LR beams with tuning local resonance. In particular, some interesting band gap phenomena observed theoretically in Refs. [30] and [32] are verified here by our experimental results.

While LR periodic structures, including LR beams considered here, are commonly proposed as a technique for vibration and wave propagation control in engineering structures, it should be noted that, in the field of vibration control engineering, the use of secondary resonators (well-known as dynamic vibration absorbers in mechanical engineering) attached on a primary structure to suppress vibrations and waves has been investigated for decades [38–41]. For the purpose of broadband applications, the concept of multiresonant damped resonators [39] as well as distributed/continuous damped resonators [38,41] has been well developed.

This paper is organized as follows. Section 2 describes the configuration of the proposed LR beam structure considered in this work as well as the theoretical background for the numerical analysis of such a structure. In Sec. 3, we first present a detailed description about the design and fabrication of the LR beam specimens involved in this work. Then, an in-depth numerical study of the LR beam specimens is followed. Three aspects are examined, including band gap behavior in infinite systems, vibration transmittance of finite structures, and the effects of damping on band gap behavior and vibration transmittance. For the last point in Sec. 3, we present experimental results on our LR beam specimens and then compare them with theoretical predictions. Finally, we conclude the paper in Sec. 4.

2 The Proposed Structure and Modeling of the Structure

2.1 The Proposed Structure. The LR beam structure considered in this work is made of a periodic array of beam-like resonators [37–39] mounted on a thin homogeneous beam, as shown in Fig. 1(a). The lattice constant (spacing between two adjacent resonators) of the periodic structure is a. The width and the thickness of the host beam are b and h, and the dimension of the beam-like resonator is 2l × w × t, as denoted in Fig. 1(a).

To simplify the numerical modeling of the LR beam structure shown in Fig. 1(a), we assume that each beam-like resonator is connected to the host beam via a lumped point. For simplicity, we also neglect the twisting moments that might be exerted on the host beam by the beam-like resonators. It is known that a beam-like resonator can produce multiple transverse resonances [38]. In this work, we are only interested in the lowest resonance. Therefore, the beam-like resonator can be treated as a mass-spring-mass system [39,42]. The designed structure can then be simplified into a more physical and accessible model, as shown in Fig. 1(b).

In the simplified model, m, and k, are the effective mass and effective stiffness of the beam-like resonator, respectively, and m is the additional lumped mass introduced into the host beam at the point where the beam-like resonator is mounted.

2.2 Characterization of the Beam-Like Resonator. The steady-state vibrational behavior of the beam-like resonator shown in Fig. 1(a) can be characterized by its driving-point
whose dynamic stiffness is simply
\[ D_t = -\omega^2 m_s \] (1)
where \( \omega \) is the circular frequency, \( m_s = \rho_s A_t l_0 \) represents the mass of such a short segment, and \( \rho_s \) and \( A_t \) are the material density and cross-sectional area of the double-ended beam.

On the other hand, the driving-point dynamic stiffness at the root of each cantilever segment, \( D_c \), can be derived in a closed form by using analytical wave solutions, as described in Ref. [37]. In what follows, we also present a detailed derivation of \( D_c \). Based on the assumption of Bernoulli–Euler bending theory, the governing equation for the cantilever beam is
\[ E_t I_t \frac{\partial^4 y(x,t)}{\partial x^4} + \rho_t A_t \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \] (2)
with the following boundary conditions:
\[ E_t I_t \frac{\partial^4 y(0,t)}{\partial x^4} = F_0 e^{i\omega t} \] (3)
\[ \frac{\partial y(0,t)}{\partial x} = 0 \] (4)
\[ \frac{\partial^2 y(l_t, t)}{\partial x^2} = 0 \] (5)
\[ E_t I_t \frac{\partial^4 y(l_t, t)}{\partial x^4} = 0 \] (6)
It is well known that the steady-state solution to Eq. (2) can be written in the form
\[ y(x, t) = Y(x)e^{i\omega t} = |C_1 \sin(\beta l_t) + C_2 \cos(\beta l_t) + C_3 \sinh(\beta l_t) + C_4 \cosh(\beta l_t)|e^{i\omega t} \] (7)
where
\[ \beta = \left( \frac{\rho_t A_t \omega^2}{E_t I_t} \right)^{1/4} \] (8)
is the flexural wavenumber of the cantilever beam.
Substituting Eq. (7) into Eqs. (3)–(6) gives
\[ C_1 = \frac{F_0}{2EI\beta^3} \] (9)
\[ C_2 = 1 - \frac{\sin(\beta l_t) \sinh(\beta l_t) + \cos(\beta l_t) \cosh(\beta l_t)}{\sin(\beta l_t) \cosh(\beta l_t) + \cos(\beta l_t) \sinh(\beta l_t)} C_1 \] (10)
\[ C_3 = -C_4 \] (11)
\[ C_4 = 1 + \frac{\sin(\beta l_t) \sinh(\beta l_t) + \cos(\beta l_t) \cosh(\beta l_t)}{\sin(\beta l_t) \cosh(\beta l_t) + \cos(\beta l_t) \sinh(\beta l_t)} \] (12)
Therefore, the driving-point dynamic stiffness at the root of the cantilever beam is given by
\[ D_c = \frac{F_0}{\omega^2} = \frac{F_0}{C_2 + C_4} = \frac{-E_t I_t (\beta l_t)^3 \sin(\beta l_t) \cosh(\beta l_t) + \cos(\beta l_t) \sinh(\beta l_t)}{l_t^2} \frac{1}{1 + \cos(\beta l_t) \cosh(\beta l_t)} \] (13)
Thus, the driving-point dynamic stiffness of the entire beam-like resonator is given by
\[ D_t = D_c + 2D_e \] (14)
Next, we explore the effective properties of the beam-like resonator. For ease of analysis, we rewrite the expression for \( D_c \) (see Eq. (13)) as
\[ D_c = \frac{-E_t I_t (\beta l_t)^3}{l_t^2} \tanh(\beta l_t) + \tan(\beta l_t)}{\cos(\beta l_t) \cosh(\beta l_t)} \] (15)
It is evident that \( D_c \) can be infinite (note that damping is excluded here) at the frequencies determined by
\[ \cos(\beta l_t) \cosh(\beta l_t) = -1 \] (16)
This equation actually represents the natural frequency equation of the cantilever beam of length \( l_t \) [43]. The lowest root is found to be [43]
\[ \beta l_t = 1.875 \] (17)
which is truncated at four decimals. By using Eq. (8), we have
\[ \omega^2 = \left( \frac{1.875}{l_t} \right)^2 \frac{E_t I_t}{\rho_t A_t} \] (18)
Moreover, it can be seen that \( D_c \) in Eq. (15) can be zero at the frequencies given by
\[ \tanh(\beta l_t) + \tan(\beta l_t) = 0 \] (19)
The lowest nonzero root of this equation is
\[ \beta l_t = 2.365 \] (20)
being also truncated at four decimals. This further gives
\[ \omega^2 = \left( \frac{2.365}{l_t} \right)^2 \frac{E_t I_t}{\rho_t A_t} \] (21)
Note that each cantilever segment can be approximated by a mass-spring-mass system to characterize its lowest resonant modes. We denote the parameters of such an approximate system as \( m_{\text{co}}, k_c, \) and \( m_c \), as shown in Fig. 3. Here, \( m_{\text{co}} \) is considered as a lumped mass attached to the host beam. The equation of motion for the lumped system shown in Fig. 3 can be written as
\[
\begin{bmatrix}
  k_c & -k_c \\
  -k_c & k_c
\end{bmatrix}
\begin{bmatrix}
  m_{\text{co}} & 0 \\
  0 & m_c
\end{bmatrix}
\begin{bmatrix}
  u_{\text{co}} \\
  u_c
\end{bmatrix}
= \begin{bmatrix}
  F_0 \\
  0
\end{bmatrix}
\] (22)
approximate system can be obtained by the following equations:

\[ D_c = \frac{F_0}{u_{c0}} = -\omega^2 m_0 + \frac{-\omega^2 k_c m_c}{k_c - \omega^2 m_c} \]  

(23)

This may be further expressed by

\[ D_c = -\omega^2 m_0 \left( \frac{m_0 + m_c}{m_0} - \frac{\omega^2}{\omega_i^2} \right) \]  

(24)

where \( \omega_i = (k_c/m_c)^{1/2} \) is a resonance frequency.

It is seen in Eq. (24) that \( D_c \) becomes infinite at the resonance frequency \( \omega_i \), and it is zero at the frequency given by

\[ \omega^2 = \omega_i^2 \left( \frac{m_0 + m_c}{m_0} \right) \]  

(25)

The approximate system should have the same characteristic frequencies (i.e., the frequencies at which the driving-point dynamic stiffness can be infinite or zero) and equal total mass as the original cantilever segment; therefore, the parameters of the approximate system can be obtained by the following equations:

\[ \omega_i^2 = \frac{k_c}{m_c} = \frac{(1.875)^4 E I_t}{\rho A_t} \]  

\[ \omega^2 = \frac{(m_0 + m_c)}{m_0} = \frac{(2.365)^4 E I_t}{\rho A_t} \]  

(26)

Solving these equations gives

\[ m_0 = 0.395\rho A_t l_t, \quad m_c = 0.605\rho A_t l_t, \quad k_c = \frac{14.953 E I_t}{l_t^2} \]  

(27)

Therefore, the parameters of the effective system (see Fig. 1(b)) of the entire beam-like resonator are as follows:

\[ m_0 = m_c + 2m_{c0} = \rho A_t l_u + 0.395\rho A_t(2l_t) \]  

\[ m_c = 2m_{c0} = 0.605\rho A_t(2l_t) \]  

\[ k_c = 2k_c = 14.953 \frac{E I_t}{l_t^2} \]  

(28)

where \( l_u + 2l_t = 2l \).

Further, the driving-point dynamic stiffness of the effective system is found to be (similar to Eq. (23))

\[ D_c = -\omega^2 m_0 + \frac{-\omega^2 k_c m_c}{k_c - \omega^2 m_c} \]  

(29)

It should be noted that the approximation represented by \( D_c \) (see Eq. (29)) is only valid at frequencies lower and somewhat higher than the characteristic frequency given by Eq. (25) according to the above derivations. In addition, \( D_c \) cannot capture the torsional effects of the original beam-like resonators that are neglected here but do exist in reality, since the torsional vibration of the beam-like resonators can be produced by the vibrating host-beam. However, fortunately, for all the LR beam specimens considered in the present work, our numerical simulations will show that the torsional effects of the beam-like resonators can indeed be neglected in the frequency range considered in this work. This aspect will be demonstrated in Sec. 3, where the numerical predictions based on the approximation given above (where torsional effects are neglected) show very good agreement with those obtained from the finite element simulations of the original structures, in which the torsional effects are fully taken into account.

**2.3 Propagation Constants of an Infinite Periodic System.** Free harmonic wave motion in a one-dimensional infinite periodic system can be characterized by the so-called propagation constant, \( \mu \) [2–4]. The real part of \( \mu \) is known as the “attenuation constant”; it quantifies the exponential decay rate of the decaying wave as it traverses a unit cell of the periodic system. The imaginary part of \( \mu \) has been called the “phase constant”; it represents the phase difference of wave motion between adjacent unit cells of the periodic system. The values of \( \mu \) always occur in positive and negative pairs, corresponding to identical but opposite-going waves.

The LR beam structure considered in this work is a typical bicoupled periodic structure. Thus, at any frequency, there are four different propagation constants (two pairs of positive and negative values) [3]. In order to find the propagation constants, a variety of analytical methods [4] can be adopted, including the receptance method [1–3], the transfer matrix method [29,30], the spectral element method (SEM) [32], and the phased array method [44]. In particular, the SEM can be considered as an appropriate reorganization of the receptance method or that of the transfer matrix method. The main benefit of applying SEM to the analysis of periodic structures is that the exact dynamic stiffness matrices (also known as the spectral element matrices) of a variety of structural components are available in an existing textbook [45], making the dynamic modeling simpler and more straightforward. By using the SEM, an explicit equation for determining propagation constants of an LR beam carrying a periodic array of resonators has been derived in Ref. [32]. In what follows, we present the main steps of the formulations of the SEM.

First, a unit cell should be taken from the LR beam for dynamic modeling. One appropriate choice for the unit cell of the LR beam is presented in Fig. 4, where the vectors \( q_L \) and \( q_R \) are the displacement vectors at the boundary of the unit cell and \( f_L \) and \( f_R \) are the loading force vectors. The subscripts “L” and “R” denote the left and right ends, respectively. Since bending beams are considered here, each boundary displacement or force vector comprises two components, i.e.,

\[ q_L = \left\{ \begin{array}{c} u_l \\ \theta_L \end{array} \right\}, \quad q_R = \left\{ \begin{array}{c} u_r \\ \theta_R \end{array} \right\}, \quad f_L = \left\{ \begin{array}{c} Q_L \\ M_L \end{array} \right\}, \quad f_R = \left\{ \begin{array}{c} Q_R \\ M_R \end{array} \right\} \]  

(30)

\[ k_t \rho, A, E, I, a \]

\[ m_t \]

\[ m_0 \]
where $\alpha$ and $\theta$ are the transverse displacement and the slope due to bending; $Q$ and $M$ are the transverse shear force and bending moment.

Under the framework of the SEM, the unit cell shown in Fig. 4 can be modeled as a single spectral beam element [45] with a lumped element attached to its left end. Then, in order to derive the equation of motion for the entire unit cell, we need to know the dynamic stiffness matrix associated with the spectral beam element and additional dynamic stiffness introduced by the attached lumped system at the left end. The latter has been obtained in Sec. 2.2, represented by the driving-point dynamic stiffness $D_r$ (see Eq. (29)), while the former can be found immediately in the textbook [45]. Under the assumption of Bernoulli–Euler bending theory, the dynamic stiffness matrix (or spectral element matrix) of the beam element is expressed by [32,45]

$$D_{\text{beam}} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

(31)

with the submatrices $D_{11}, D_{12}, D_{21},$ and $D_{22}$ given by [32,45]

$$D_{11} = \frac{EI}{a^3} \begin{bmatrix} p & \gamma a \\ \gamma a & qa^2 \end{bmatrix}, \quad D_{12} = \frac{EI}{a^3} \begin{bmatrix} -p & \gamma a \\ -\gamma a & qa^2 \end{bmatrix},$$

$$D_{22} = \frac{EI}{a^3} \begin{bmatrix} p & -\gamma a \\ -\gamma a & qa^2 \end{bmatrix}$$

(32)

where

$$p = \frac{\cos(k_0 a) + \sin(k_0 a) \cos(k_0 a) - \sin(k_0 a)}{a}, \quad \rho = \frac{\sin(k_0 a) + \cos(k_0 a) - \cos(k_0 a)}{a},$$

$$q = \frac{-\sin(k_0 a) + \sin(k_0 a) \cos(k_0 a) - \cos(k_0 a)}{a}, \quad \phi = \frac{-\sin(k_0 a) + \cos(k_0 a) \cos(k_0 a)}{a},$$

$$\gamma = \frac{-\cos(k_0 a) \sin(k_0 a) - \sin(k_0 a) \sin(k_0 a)}{a}, \quad \Delta = 1 - \cos(k_0 a) \cos(k_0 a)$$

In the above, $k_0 = \left(\rho a \sigma / EI\right)^{1/3}$ denotes the flexural wavenumber of the beam.

Therefore, the equation of motion at a frequency $\omega$ for the unit cell is obtained as

$$\begin{bmatrix} D_{LL} & D_{LR} \\ D_{RL} & D_{RR} \end{bmatrix} \begin{bmatrix} q_L \\ q_R \end{bmatrix} = \begin{bmatrix} f_L \\ f_R \end{bmatrix}$$

(34)

where

$$\begin{bmatrix} D_{LL} & D_{LR} \\ D_{RL} & D_{RR} \end{bmatrix} = \begin{bmatrix} D_{11} & D_t \\ D_{21} & D_{22} \end{bmatrix}, \quad D_t = \begin{bmatrix} D_r \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(35)

Since wave propagation in an infinite periodic system is considered here, the displacement and force vectors associated with the two boundaries of the unit cell can be related by using Bloch theorem [46],

$$q_k = e^{i\mu k} q_k, \quad f_k = -e^{i\mu k} f_k$$

(36)

where $\mu$ is the propagation constant [2,4].

Combining Eqs. (34) and (36) leads to the following quadratic eigenvalue problem for $e^\mu$:

$$\begin{bmatrix} [D_{RL} + (D_{LL} + D_{RR})e^\mu + D_{LR}e^{2\mu}] & 0 \\ 0 & [D_{RL} + (D_{LL} + D_{RR})e^\mu + D_{LR}e^{2\mu}] \end{bmatrix} \begin{bmatrix} q_L \\ q_R \end{bmatrix} = 0$$

(37)

Nontrivial solutions of Eq. (37) are only obtained provided

$$[D_{RL} + (D_{LL} + D_{RR})e^\mu + D_{LR}e^{2\mu}] = 0$$

(38)

which, on expansion, further gives

$$\cosh^2 \mu + z_1 \cosh \mu + z_2 = 0$$

(39)

with

$$z_1 = -\frac{\cos(k_0 a) + \cos(k_0 a) - \sin(k_0 a)}{a}, \quad z_2 = \frac{\sin(k_0 a) \cos(k_0 a) - \cos(k_0 a) \sin(k_0 a)}{a}$$

(40)

where $D_r$ represents the driving-point dynamic stiffness of the attached mass-spring-mass system, whose expression is given by Eq. (29). If we consider the original beam-like structure as the attachment and neglect its rotational effects, the corresponding propagation constants can also be calculated using Eq. (39), but the term $D_r$ should be replaced by $D_e$, which is given by Eq. (14).

For a given frequency, Eq. (39) determines two pairs of propagation constants ($\pm\mu_1, \pm\mu_2$) with possibly complex values. The relation between the propagation constants and the frequencies represents the “dispersion characteristics” for the harmonic wave propagation in an infinite periodic LR beam. The plot of propagation constants versus frequencies constitutes the well-known frequency band structure, in which the “pass bands” and “band gaps” of wave motion can be identified. In a band gap frequency range, all propagation constants have a nonzero real part; thus, all characteristic waves will be attenuated when traveling along the infinite system, while in a pass band frequency range, at least one pair of propagation constants have a zero real part; hence, at least one type of characteristic wave (positive-going or negative-going) can propagate through the infinite system without attenuation.

2.4 Frequency Response of a Finite Structure. Frequency response function (FRF) of a finite periodic structure has commonly been used to verify the predictions for an infinite periodic system. It is also always employed to examine the vibration reduction performance of a finite periodic structure, due to its band gap effects.

In principle, the FRF of finite LR beams could be simply calculated by the traditional transfer matrix method through a chain product of the transfer matrices of individual structural elements. However, numerical ill-conditioning and inefficiency may occur in the computational process. Such a problem of transfer matrix method has been mentioned in a number of previous works on periodic structures [4,12,28,47]. Improvements of the transfer matrix method do exist [12,28]. However, considerable additional treatments are required.

In this work, we use the SEM [45] to calculate the FRF of finite LR beams. The SEM represents a good compromise between the large number of elements required in the conventional finite element method and the numerical problems associated with the transfer matrix method [47]. In what follows, we present a brief introduction to how to employ the SEM to calculate the FRF of a finite LR beam with the configuration shown in Fig. 5(a), which represents a general finite structure corresponding to the system shown in Fig. 1(b). In Fig. 5(a), $N$ denotes the number of attached resonators, $a_0$ describes the location of the first resonator from the left end of the host beam, and $\theta_0$ denotes the location of the last resonator from the right end. The finite host beam is assumed to have free-free boundary conditions and to be excited by a harmonic transverse displacement excitation at the left end, represented by $u_0 e^{i\omega t}$. Assume that our purpose is to calculate the transverse frequency response $u_0 e^{i\omega t}$ at the right end.

Under the framework of the SEM, each beam segment without discontinuities can be modeled as a spectral beam element [45]. Therefore, the host beam shown in Fig. 5(a) can be modeled as a combination of $N + 1$ spectral beam elements with $N + 2$ nodes. The numberings for the elements and nodes are depicted in Fig. 5(b).
The dynamical effect of each spectral beam element can be characterized by a dynamic stiffness matrix given by

$$D_{\text{beam}}^{(e)} = \begin{bmatrix} D_{11}^{(e)} & D_{12}^{(e)} \\ D_{21}^{(e)} & D_{22}^{(e)} \end{bmatrix}$$ (41)

where the superscript “e” denotes the numbering of element. The explicit expressions of the submatrices of Eq. (41) are the same as those described by Eqs. (32) and (33), except that the lattice constant term $a$ involved should be replaced by the associated element length $l_{\text{beam}}^{(e)}$. From Fig. 5, we can see that

$$f_{\text{beam}}^{(e)} = \begin{cases} a_l, & e = 1 \\ a_i, & e = 2, 3, \ldots, N \\ a_R, & e = N + 1 \end{cases}$$ (42)

Moreover, the dynamic stiffness introduced by each attached mass-spring-mass system can be represented by its driving-point dynamic stiffness $D_i$ (see Eq. (29)). Thus, they can be considered as lumped elements whose dynamic effects can be characterized by the following dynamic stiffness matrix:

$$D_i = \begin{bmatrix} D_i & 0 \\ 0 & 0 \end{bmatrix}$$ (43)

which has been presented in Eq. (35).

The global dynamic stiffness matrix for the entire finite LR beam can be obtained by assembling the matrices of all the individual beam elements and lumped elements, according to the standard procedure of the conventional finite element method. The global dynamic stiffness matrix is written as

$$D = \begin{bmatrix} D_1^{(1)} & D_1^{(2)} & \cdots & 0 \\ D_1^{(2)} & D_1^{(3)} & \cdots & D_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & D_N^{(1)} & \cdots & D_N^{(N)} \end{bmatrix}$$ (44)

Therefore, the equation of motion of the entire finite LR beam is

$$Dq = f$$ (45)

with

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{N+2} \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_i \\ \vdots \\ f_{N+2} \end{bmatrix}, \quad q_i = \begin{bmatrix} u_i \\ \theta_i \end{bmatrix},$$ (46)

$$q_{N+2} = \begin{bmatrix} u_h \\ \theta_h \end{bmatrix}, \quad f_1 = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}.$$ (46)

If the excitation $u_i$ is assigned, the frequency response at each node (see Fig. 5(b)), including that at the right end (i.e., $u_n$) can be readily calculated by Eq. (45). Then the vibration reduction performance of the finite LR beam can be characterized by the transverse vibration transmittance [32] given by

$$T = \begin{bmatrix} u_n \\ u_i \end{bmatrix}$$ (47)

which can also be regarded as the amplitude of the normalized response $u_n$ with respect to the excitation $u_i$.

3 Numerical and Experimental Results

3.1 Specimens and Experimental Setup. We fabricated five specimens of LR beams using the same host beam but five different types of beam-like resonators. All these LR beam specimens feature the same lattice constant: $a = 90$ mm. The host beam and the beam-like resonators are all made of aluminum (Young’s modulus $E = 7 \times 10^{10}$ Pa, density $\rho = 2700$ kg/m$^3$, Poisson’s ratio $\nu = 0.33$). The geometric properties of the host beam are: total length $L = 1.5$ m, width $b = 20$ mm, and thickness $h = 3.8$ mm. Each LR beam specimen is fabricated by mounting 16 same-type beam-like resonators along the host beam with identical spacing. The distance between the first beam-like resonator and the left end of the host beam is $a_l = 75$ mm, and it follows that the
distance between the last resonator and the right end is \( a_q = 75 \text{ mm} \), since the resonator spacing is 90 mm and the total length is 1.5 m.

It should be mentioned that, at the resonator-attachment points of the host beam as well as the center point of each beam-like resonator, small holes are drilled through their thickness. The mounting of the beam-like resonators is thereby realized by inserting a bolt through the holes of the beam-like resonator, the washer, and the host beam and securing it by a mated nut, as illustrated in Fig. 1(a). Since the nut is tightened by applying torque, some pretension will be introduced into the bolts as well as some local parts of the host beam and the beam-like resonators. Such pretension may affect the physical properties (e.g., the damping) of the LR beams but is neglected in our numerical simulations.

The five types of beam-like resonators used for the construction of different LR beam specimens are designed such that their effective systems (see Fig. 1(b)) have identical total mass, \( m_qA_2l_r \), and have different resonance frequency, \( f_i \) (\( = \omega_i/2\pi \)). From Eq. (26), we know that

\[
\frac{\omega_i^2}{\rho_t} = \frac{(1.875)^4 E_i l_i}{\rho_t A_i} = \frac{(1.875)^4 E_i t_i^2}{2 \rho_t l_r},
\]

which suggests that the resonance frequency \( f_i \) (\( = \omega_i/2\pi \)) can be tuned by varying the cantilever-beam length \( l_i \) and/or the thickness \( t \). In our design, the thickness of the beam-like resonators is fixed to \( t = 2 \text{ mm} \) and the washers used are all identical, whose effective supporting dimension along the length of the beam-like resonator is \( l_w = 3 \text{ mm} \). Then, the relation between the cantilever-beam length \( l_i \) and the entire-beam length \( (2l) \) is simply

\[
l_i = \frac{2l - l_w \rho_t}{2}
\]

Therefore, from Eq. (48), we can find that the resonance frequency \( f_i \) can be tuned by adjusting the entire-beam length \( (2l) \) according to

\[
f_i = \frac{1}{2\pi} \sqrt{\frac{(2 \times 1.875)^4 E_i t_i^2}{(2l - l_w)^2} 12 \rho_t}
\]

On the other hand, remember that our designs also require that different types of beam-like resonator should have the same total mass; \( \rho_t A_2l_r = \rho_t(w \times 2l) \), where \( S_i \) represents the surface area. Since the thickness \( t \) is fixed, the surface area \( S_i \) should also be fixed to ensure an identical total mass \( \rho_t S_i \). Then, in the designs, if the entire-beam length \( (2l) \) is adjusted, the width \( w \) should be adjusted simultaneously according to

\[
w = \frac{S_i}{2l}
\]

where \( S_i \) is the fixed surface area used in the design. In this work, we choose \( S_i = 1440 \text{ mm}^2 \).

The geometrical and physical parameters of the designed and fabricated beam-like resonators are listed in Table 1. The length of the five types of beam-like resonators is adjusted from 95 mm to 75 mm, and thus their resonance frequencies are tuned from 777 Hz to 1269 Hz, which can be predicted by Eq. (50). The widths of the beam-like resonators, \( w \), are designed following Eq. (51), but they are rounded as integral millimeters for ease of fabrication. The parameters of the effective systems (i.e., \( m_q, m_r \), and \( k_r \)) are predicted by Eq. (28). It can be seen that the effective systems have almost identical effective masses \( (m_q, m_r) \) but different effective stiffness \( (k_r) \).

In order to demonstrate the validity of the approximation of a beam-like resonator as an effective lumped mass-spring-mass system, Fig. 6 shows a comparison between the exact and approximate values of the driving-point dynamic stiffness for the first beam-like resonator \( (f_i = 777 \text{ Hz}) \) specified in Table 1. The exact values are predicted by Eq. (14), while the approximate values are calculated by Eq. (29). It is clearly shown that the latter are very close to the former, especially in a low frequency range. As for other beam-like resonators listed in Table 1, the agreements between their approximate and exact models are generally better than the one considered in Fig. 6, if considering in the same frequency range. This is because the length of other beam-like resonators is shorter than the first one, and their second resonances exist in a higher frequency range. Actually, in the following subsection, our numerical and experimental results are all considered in a frequency range below 2000 Hz. In such a frequency range, the approximate models (featuring the lumped parameters listed in Table 1) of the beam-like resonators generally match very well with their exact counterparts.

Figure 7 shows the experimental setup used to measure the vibration transmittance FRFs of the LR beam specimens. Such a test scheme has been commonly adopted in earlier experimental works on LR structures [27,29]. The specimen is hung up through two soft rubber threads so that free-free boundary condition is suggested. In the experiments, a random signal up to 6.4 kHz is produced by the B&K pulse system and amplified by the B&K power amplifier 2732. Then, the magnified signal is fed to the B&K vibration exciter 4824 to generate vibrations at the left end.
Fig. 8 (a) Schematic and (b) photograph of the experimental setup

of the host beam. The vibration acceleration responses at the left end (referred to as an input point) and the right end (considered as an output point) of the host beam are measured by two B&K accelerometers 4507B. We assume the measured acceleration FRFs at these two points are represented by \( \ddot{u}_i(\omega) \) and \( \ddot{u}_o(\omega) \), respectively. Thus, the vibration transmittance FRF is obtained as

\[
T(\omega) = \frac{\ddot{u}_o(\omega)}{\ddot{u}_i(\omega)} = \left| \frac{-\omega^2 \ddot{u}_o(\omega)}{-\omega^2 \ddot{u}_i(\omega)} \right| = \left| \frac{\ddot{u}_i(\omega)}{\ddot{u}_i(\omega)} \right|
\]

(52)

3.2 Numerical Study. This subsection is devoted to a numerical study of the LR beam structures mentioned above. The purpose is twofold: (1) to test the methodology presented in this paper and (2) to illustrate some band gap characteristics of the LR beams under consideration.

3.2.1 Band Gaps in Infinite Systems. First, we examine the band gaps in infinite LR beams. As an example, Fig. 8 shows the calculated propagation constants and thus the complex band structure for the case of LR beam with resonators tuned at \( f_l = 869 \) Hz, whose parameters are given in Subsection 3.1. In Fig. 8(a), the predictions based on the exact model, in which the input dynamic stiffness of resonators is represented by \( D_i \) (see Eq. (14)), are compared with those based on the approximate model, for which the input dynamic stiffness is described by \( D_i \) (see Eq. (29)). As expected, the approximate model matches very well with the exact model in the frequency range considered. In addition, Fig. 8(b) shows a closer view of the results within the frequency range 900–960 Hz.

Although two pairs of propagation constants \((\pm \mu_1, \pm \mu_2)\) can be obtained at each given frequency, only the positive values are presented in Fig. 8. According to the physical properties of the solution of propagation constants, different types of frequency bands can be distinguished on the frequency axis [32]. A propagation-attenuation (PA) band represents a frequency range where one pair of the propagation constants is purely imaginary, while the other pair is of the form \( \mu = \Re(\mu) + i\pi \) with \( n \) integer; a propagation-propagation (PP) band, however, is referred to as a frequency range where both the pairs of the propagation constants are purely imaginary; an attenuation-attenuation (AA) band represents a frequency range where both the pairs of the propagation constants has the form \( \mu = \Re(\mu) + i\pi \) with \( n \) integer; and a complex (C) band is considered to be a frequency range that the propagation constants exist in complex conjugate pairs (i.e., \( \mu_2 = \mu_1^* \)) or, equivalently, \( \Re(\mu_2) = \Re(\mu_1) \) and \( \Im(\mu_2) = -\Im(\mu_1) \).

The AA and C bands are of the most interest in this work, since in these bands no purely imaginary propagation constants exists, and therefore all waves will be attenuated due to the nonzero \( \Re(\mu) \). Actually, such frequency bands are considered as the so-called band gaps (or stop bands). Figure 8 indicates that the LR beam exhibits four types of frequency bands in the frequency range considered, and two band gaps are formed. The first AA band and the C band constitute the first band gap, while the second AA band defines the second band gap. It should be noted that the lower edge frequency of the second band gap (the second AA band) represents the lowest Bragg frequency \( f_B \) associated with the periodic lattice. Such a frequency can be determined by the lowest Bragg condition \((k_{sd}/\pi = 1)\), which gives

\[
f_B = \frac{1}{2\pi} \left( \frac{1}{a} \right)^2 \sqrt{\frac{EI}{pa}} = 1083Hz
\]

(53)

It can be seen that the Bragg frequency \( f_B \) is dependent on the lattice constant \( a \) (i.e., the spatial periodicity) but has no reference to the local resonance. In addition, at the Bragg frequency \( f_B \), the lattice constant is equal to the half wavelength of the flexural waves in the host beam.
It is known that, for periodic flexural beams, the wave attenuation performance inside band gaps can be well quantified by the smaller attenuation constant, since it characterizes the least rapidly decaying wave that carries energy the farthest [31,32]. The smaller attenuation constant is defined such that

$$\min \left\{ \Re \left( \mu_1 \right) \right\} = \min \left\{ \left| \Re \left( \mu_1 \right) \right|, \left| \Re \left( \mu_2 \right) \right| \right\}$$  \hspace{1cm} (54)$$

The plots of the smaller attenuation constant for the LR beam considered in Fig. 8 is shown in Fig. 9, in which band gaps can be easily identified as frequency regions where the smaller attenuation constant is nonzero, which implies that no freely propagating waves exist. The plots of the smaller attenuation constant clearly indicate the location, width, and wave attenuation performance of the band gaps. The two band gaps identified in Fig. 9 originate from different physical mechanisms (i.e., the LR mechanism or Bragg scattering mechanism) [28,30,31]. The first band gap can be considered as an LR band gap, while the second band gap should be classified as a Bragg band gap, since the location of the first band gap can be significantly affected by tuning the resonance frequency of the resonators, but the position of the second band gap remains around the Bragg frequency $f_B$.

It is of interest to note that the horizontal dashed line marked at $2l = 2l_0$ in Fig. 10 denotes a condition for the transition between LR and Bragg band gaps. It can be seen, when $2l > 2l_0$, the first band gap is a LR band gap, while the second one a Bragg band gap. When $2l < 2l_0$, however, the first band gap becomes a Bragg band gap, but the second one comes to be an LR band gap. Therefore, the band gap transition takes place at $2l = 2l_0$.

The band gap transition phenomenon in LR beams has been observed and discussed by Liu and Hussein [30]. They establish a criterion for the band gap transition by examining the evolution of the pass band that appears between the two band gaps (i.e., the LR and Bragg band gaps) and express the criterion in terms of group velocity gradient associated with this pass band, as well as in terms of the minimum frequency of this pass band [30]. More recently, Xiao et al. [32] further studied the band gap transition phenomenon in LR beams. They show that the band gap transition state can alternatively be identified by examining the evolution of band gaps, which are characterized on the basis of the attenuation constants. Furthermore, they derive explicit formulas for a direct prediction of the conditions for the band gap transition [32].
way of examining band gap transition phenomena in this work follows that of Ref. [32].

In Fig. 10, besides the band gap transition state, another interesting situation is denoted by the horizontal dotted line at $2f = 2f_{\text{LR}}$. Such a special situation is referred to as a band gap near-coupling state in Ref. [32]. Under this situation, the width of the pass band located between the LR and Bragg band gaps becomes the narrowest. Since the width of this pass band is much smaller than those of the band gaps, it seems as if a superwide "pseudo band gap" is formed by a combination of the LR and Bragg band gaps. Such a band gap near-coupling phenomenon in LR beams (bicoupled periodic system) is similar to the exact-coupling phenomenon observed in some LR monocoupled periodic systems (e.g., LR string/rod [28, 36]), in which the width of the pass band located between the LR and Bragg band gaps can be tuned to zero.

A further interesting feature that should be drawn from Fig. 10(a) is that the resonance frequency $f_1$ does not necessarily lie in an LR band gap. Figure 10(a) shows that, when $2f < 2f_{\text{LR}}$, the dash-dotted curve, which represents the evolution of resonance frequency $f_1$, may pass through the Bragg band gap region and even the pass band region. This suggests that the resonance frequency may lie in a Bragg band gap or in a pass band. This can be understood by the solution of attenuation constants at the resonance frequency. In general, at the resonance frequency $f_1$, only one pair of attenuation constants (denoted as $\pm \text{Re}(\mu_1)$) is infinite, while the other pair (denoted as $\pm \text{Re}(\mu_2)$) always has a finite value, with the possibility of being zero. As an example, the attenuation constants for the case of LR beam with resonators tuned at $f_1 = 1269 \text{ Hz}$ ($2f_1 = 0.075 \text{ m}$) are depicted in Fig. 11. It is clearly seen that, at the resonance frequency $f_1 = 1269 \text{ Hz}$, Re($\mu_2$) $\to \infty$, while Re($\mu_1$) $= 0$. Thus, in this case, the resonance frequency of local resonators actually lies within a pass band.

In order to achieve a more feasible understanding of the formation of band gap maps shown in Fig. 10, in what follows, we perform an examination of the dependence of $2f$ upon the band edge frequencies associated with the LR beams. Here, the band edge frequencies refer to the frequencies separating different types of frequency bands (i.e., PA, PP, AA, and C bands). Following the derivations presented in Ref. [32], it is known that the band edge frequencies of noncomplex bands (i.e., PA, PP, and AA bands) are governed by the following four equations:

$$\cos(k_o \alpha) = \pm 1$$  \hspace{1cm} (57)

$$D_t = 4EI k_o^4 \left( \tanh \frac{k_o \alpha}{2} - \tan \frac{k_o \alpha}{2} \right)^{-1}$$  \hspace{1cm} (58)

$$\text{Re} \left( k_{5,6} \right) = \frac{\sqrt{\sin(k_o \alpha) \pm \sqrt{\sin(k_o \alpha)}}}{\sin(k_o \alpha)}$$  \hspace{1cm} (59)

which, for ease of reference, are referred to as $r_{5,6}$ (Eq. (55)), $r_B$ (Eq. (56)), $r_6$ (Eq. (57)), and $r_{5,6}$ (Eq. (58)) frequencies, respectively. Note that the $r_{5,6}$ and $r_B$ frequencies actually represent the odd ($k_o \alpha = 1, 3, 5, ...$) and even ($k_o \alpha = 2, 4, 6, ...$) order Bragg frequencies, respectively. The lowest $r_B$ frequency is simply the Bragg frequency $f_B$ given by Eq. (53).

Reference [32] also shows that the band edge frequencies of the complex bands (i.e., C bands) are given by

$$D_t = 4EI k_o^4 \left[ \cos(k_o \alpha) - \cos(k_o \alpha) \right]$$  \hspace{1cm} (59)

where $k_o \alpha \in [2n\pi, (2n+1)\pi]$, and such band edge frequencies are referred to as $p$ frequencies for ease of reference.

Note that the term $D_t$ involved in Eqs. (56), (58), and (59) is given by Eq. (14); it can be considered as a function of $2f$. Thus, Eqs. (56), (58), and (59) actually determine the relations between $2f$ and the band edge frequencies or, in other words, the dependences of $2f$ upon the band edge frequencies. However, such dependences may not be found analytically, due to the complexity of the

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**Fig. 10** (a) Evolution of band gap behavior with changing total length of the beam-like resonator: $2l$. (b) Closer view of the area denoted by the pane in (a).

**Fig. 11** Attenuation constants for the LR beam with resonators tuned at $f_1 = 1269 \text{ Hz}$

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Thus, the \( D_t \) frequencies are, respectively, approximated by the following three equations:

\[
D_t = \bar{\nu}_f \\
D_t = \bar{\nu}_i \\
D_t = \bar{\nu}_p
\]  

(66) \hspace{1cm} (67) \hspace{1cm} (68)

Substituting Eq. (60) into Eqs. (66)–(68) gives

\[
\begin{align*}
\rho_R : 2l &= 2l_w + \left[ \frac{16.48 \times E_t l^2}{\rho t^2} \left( 1 + \frac{\omega^2}{\omega^2 m_l + \omega^2 m_t} \right) \right]^{1/4} \\
\rho_S : 2l &= 2l_w + \left[ \frac{16.48 \times E_t l^2}{\rho t^2} \left( 1 + \frac{\omega^2}{\omega^2 m_t + \omega^2 m_l} \right) \right]^{1/4} \\
P : 2l &= 2l_w + \left[ \frac{16.48 \times E_t l^2}{\rho t^2} \left( 1 + \frac{\omega^2}{\omega^2 m_t + \omega^2 m_l + \omega p} \right) \right]^{1/4}
\end{align*}
\]

(69) \hspace{1cm} (70) \hspace{1cm} (71)

where \( m_0 \) and \( m_t \) are approximated as fixed constants (see Eq. (62)).

Figure 12 shows the \( \rho_R \), \( \rho_S \), and \( P \) band edge frequency curves calculated analytically using Eqs. (69)–(71), as well as the \( \rho_R \) band edge frequency curve located at the Bragg frequency \( f_B \). As expected, these band edge frequency curves indicate a good capture of the band gap feature shown in Fig. 10. It is seen that the AA and C band areas marked in Fig. 12 are in good agreement with the band gap regions shown in Fig. 10. This suggests that the approximate but analytical band edge frequency equations can be employed as an efficient initial design tool. In particular, the comparison between Fig. 10(b) and Fig. 12(b) suggests that the condition for the band gap near-coupling can be approximately identified by the intersection points of the \( \rho_R \) (\( f = f_B \)) and \( \rho_S \) curves. The location of such an intersection point can be predicted analytically by using Eq. (70). Thus, we obtain the following approximate formula to enable the prediction of the condition for band gap near-coupling:

\[
2l = 2l_w \approx l_w + \left[ \frac{16.48 \times E_t l^2}{\rho t^2} \left( 1 + \frac{\omega^2 m_t}{\omega^2 m_l + \omega^2} \right) \right]^{1/4} \bigg|_{f = f_B} \tag{72}
\]

which immediately gives \( 2l_w \approx 0.0844 \text{ m} \), being very close to the true condition observed in Fig. 10(b).

### 3.2.2 Vibration Transmittance FRFs of Finite Structures.

Now we examine the vibration transmittance FRFs of finite LR beams. Consider the case of an LR beam with resonators tuned at \( f_t = 869 \text{ Hz} \) as an example. We first perform the calculation using the analytical SEM formulated in Sec. 2.4. The structural model adopted for the calculation is shown in Fig. 5, but we specifically choose \( N = 16 \) and \( a_t = a_R = 75 \text{ mm} \), according to our fabrication of the LR beam specimen, which is illustrated in Fig. 7(a). In order to validate the predictions of SEM, we further conduct a numerical simulation of the finite LR beam using the conventional
finite element method (FEM), implemented in the well-known FE package: MSC. Nastran. The structural model adopted for the FE simulation is shown in Fig. 13. The beam-like resonator is treated as a combination of a double-ended beam (of total length \(2l\)), whose midspan point is rigidly attached to the host beam and a lumped mass \(m_L\), which represents the short constrained segment of the original beam-like resonator.

The results calculated using these two methods are compared in Fig. 14. It is seen that the results obtained by the SEM agree very well with the FEM; only a little discrepancy can be found in the high frequency range. This comparison also implies that the twisting moments exerted on the host beam by the beam-like resonators (due to their torsional vibration), which are included in the SEM but neglected in the FEM, have little influence on the vibration transmittance of the finite LR beam in the frequency range considered. This is because, on the one hand, the lowest torsional mode of the beam-like resonators exists in a much higher frequency range than 2000 Hz. On the other hand, the twisting moment produced by the beam-like resonators is very small, in a frequency range well below the lowest torsional resonance.

The vertical dotted lines in Fig. 14 denote the edges of the two band gaps predicted in Fig. 13. It is seen that a significant drop on the FRF can be observed within the LR band gap range, and the profile of the drop is in good agreement with that of the smaller attenuation constant plot shown in Fig. 9. However, within the Bragg band gap region, no significant drop on the FRF can be seen. But a small drop exists around the lower gap edge (i.e., at the Bragg frequency \(f_b\)). Moreover, it is surprising that a significant peak arises inside the Bragg band gap. Such transmittance peak inside a band gap is unwanted in our design.

In order to explore more design possibilities, we further study the influence of the structural length and the global resonator location on the vibration transmittance of the finite LR beam with resonators tuned at \(f_t = 869\) Hz. Figure 16(a) shows the influence of the structural length, which is adjusted by changing the number of resonators \(N\), but retaining the lattice constant \(a\) (= 90 mm), and the choices of \(a_L (= 75\) mm) and \(a_R (= 75\) mm). It is shown that the transmittance drops inside the two band gaps generally increase with increasing structural length (number of resonators).

Of particular interest in Fig. 16(a) is the evolution of the transmittance peak inside the Bragg band gap. First, we observe that the frequency of the transmittance peak inside the Bragg gap is insensitive to the structural length, especially if the structural length is long enough. This is different from the frequencies of the regular
pass band transmittance peaks, which experience clear shift as the structural length is changed. Second, it is seen that the amplitude of the transmittance peak inside the Bragg band gap drops significantly with increasing structural length, while the amplitudes of all the regular pass band peaks do not undergo any remarkable drops. Such features of the transmittance peak inside the Bragg band gap observed here are very similar to the so-called band gap resonances discussed by Davis et al. [48]. They calculated and measured the transfer-receptance FRF of a finite periodic beam structure composed of alternating layers of materials and observed the existence of resonance peaks in the FRF within band gap frequency ranges of the corresponding infinite periodic structure. They referred to such resonance peaks in the transfer-receptance FRF as band gap resonances and demonstrated that, as the number of unit cells of the finite periodic structure is increased, the vibration response corresponding to band gap resonances (i) does not shift in frequency and (ii) drops in amplitude [48].

Figure 16(b) shows the influence of the global resonator location on the vibration transmittance of the finite LR beam. Here, the global resonator location refers to the relative position between the periodic resonator array and the host beam. Thus, it can be characterized by the parameter \( a_b \). In Fig. 16(b), two cases with different choices for the global resonator location are compared. The first case (solid line) corresponds to the system examined in Fig. 14 (solid line), while the second case (dashed line) results from the first case experiencing a rightward 30-mm translation of the resonator array. It can be seen in Fig. 16(b) that, for the second case, no transmittance peak exists inside the Bragg band gap, and the drop on the FRF shows a good capture of the band gap resonances discussed by Davis et al. [48]. They calculated and measured the transfer-receptance FRF of a finite periodic beam structure composed of alternating layers of materials and observed the existence of resonance peaks in the FRF within band gap frequency ranges of the corresponding infinite periodic structure. They referred to such resonance peaks in the transfer-receptance FRF as band gap resonances and demonstrated that, as the number of unit cells of the finite periodic structure is increased, the vibration response corresponding to band gap resonances (i) does not shift in frequency and (ii) drops in amplitude [48].

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3.2.3 Effects of Damping. As a last point, we examine the effects of damping on the behavior of both infinite and finite LR beams. This aspect is of practical interest since, on the one hand, damping exists in all real structures; on the other hand, damping can be introduced as an additional degree of freedom for the design of LR beams. We address the effects of the host-beam damping and the resonator damping separately. Here, the damping is introduced by means of complex Young’s modulus (i.e., \( E \rightarrow E(1 + j\eta_b) \) and \( E \rightarrow E(1 + j\eta_r) \), where \( \eta_b \) and \( \eta_r \) denote loss factors. It should be pointed out that the mechanical mountings for the beam-like resonators may also introduce significant damping into the structure. However, such damping might not be described in a straightforward way; thus, it is not addressed here.

Figure 17(a) shows the effects of the host-beam damping on the band gap behavior (upper panel) and vibration transmittance (lower panel) of the LR beam with resonators tuned at \( f_r = 869 \) Hz, while Fig. 17(b) depicts the effects of the resonator damping. Figure 17(a) indicates that the host-beam damping has little influence on the wave attenuation performance inside the band gaps but can exert noticeable influence on the performance within the pass bands and at the band gap edges. This is because, inside the band gaps, only a short length of the host beam can vibrate prominently, due to the large attenuation performance of the band gaps; thus, the damping of the host beam is negligible. In contrast, in the pass bands or at the band gap edges, the host beam can vibrate significantly over a very long extent; thus, the host-beam damping can exert considerable influence. Figure 17(b) shows that the resonator damping can affect the behavior of the LR band gap in a remarkable way. First, the amount of the maximum attenuation inside the LR band gap decreases significantly with increasing resonator damping. Second, the LR band gap can be broadened a lot as the resonator damping is increased. Similar effects have been previously observed in a spring-mass chain with periodically attached damped resonators [7]. The upper panel of Fig. 17(b) also indicates that the resonator damping has little influence on the wave attenuation performance at the Bragg frequency \( f_B \), which is different with the effect of the host-beam damping at the same frequency shown in the upper panel of Fig. 17(a). This is expected because the steady-state motion of the infinite periodic system at the Bragg frequency is such that [32] the localized resonators do not vibrate, while each host-beam segment between two adjacent resonators vibrates in a standing wave form.

3.3 Experimental Results. The experimental results for various LR beam specimens are presented in this subsection and they are compared with the theoretical predictions. The upper panels of Figs. 18–22 shows the experimental and numerical vibration transmittance FRFs for the five LR beam specimens mentioned in Sec. 3.1. The numerical results calculated using the SEM and using the conventional FEM are both presented for comparison. Note that the structural model used in the SEM is illustrated in Fig. 5, where the beam-like resonator is modeled as an approximate mass-spring-mass system, while the model adopted in the FEM is shown in Fig. 13, in which the beam-like resonator is treated as a combination of a double-ended beam (of total length \( 2L \)) and a lumped mass \( m_r \), representing the short constrained segment of the original beam-like resonator. It is seen that, for each case of Figs. 18–22, the results obtained using the SEM are in
very good agreement with the FEM, except some small discrepancies at high frequencies.

The lower panels of Figs. 18–22 depict the calculated smaller attenuation constants corresponding to the infinite LR beams. The predictions following the exact model, in which the dynamic stiffness of resonators is given by $D_r$ (see Eq. (14)), are compared with those based on the approximate model, in which the dynamic stiffness is represented by $D_r$ (see Eq. (29)). It can be seen in each case that the approximate model matches very well with the exact model in the frequency range considered.

In the upper panels of Figs. 18–22, we can see that the experimental FRFs show basically good agreement with the numerical FRFs in respect to the locations of the low-frequency transmittance peaks and in respect to the frequency ranges exhibiting noticeable transmittance drops due to the band gap effects. Nevertheless, significant discrepancy can be found on the amount of vibration attenuation within the LR band gaps. The maximum amount of vibration attenuation observed on the experimental results is always less than 60 dB, whereas the numerical predictions always exceed this value to a great extent. One reason for this discrepancy is that, at the frequencies of great vibration attenuation, the vibration level at the right end of the finite structure (output point) is too small to be accurately measured by the accelerometer; thus, the amount of vibration attenuation is underestimated. Other possible reasons include (but are not limited to): (1) damping effects, which are not included in the numerical simulation here; (2) periodicity perturbation due to fabrication of the specimens; (3) structural uncertainties and nonlinearities; and (4) response of other vibrational modes (e.g., axial and torsional modes) of the structure.

Figure 21(a) shows that there exists a noticeable discrepancy between the experimental and numerical FRFs in respect of the
location and amplitude of the transmittance peak inside the LR band gap. A numerical parameter study showed that this transmittance peak is highly sensitive to the placement of the load. As an illustration, two cases with different placement of the load are compared in Fig. 23, where the solid line represents the same numerical case considered in Fig. 21(a), while the dotted line corresponds to a case that the placement of the load is 6 mm from the left extreme end. The comparison shows that the placement of the load has a striking influence on the transmittance peak inside the LR band gap but much less influence on the regular pass band transmittance peaks. It should be noticed that, in our experiments, the shaker has a definite dimension and it is connected to the beam via a circular area with a radius of 6 mm, whereas in our numerical model, for ease of modeling, the load is described by a point excitation placed at the left extreme end of the structure. Therefore, the discrepancy concerning the transmittance peak inside the LR band gap observed in Fig. 21(a) may be attributed to the simplified treatment of the load in our numerical simulations.

By comparing the upper and lower panels of Figs. 18–22, it is clearly seen that the transmittance drops observed from the calculated and measured FRFs generally verify the existence of LR and Bragg band gaps in these LR beams, except that, for the first case (see Fig. 18), the attenuation performance of the Bragg band gap is too small to be revealed by a small-size finite structure, and for the second case (see Fig. 19), the Bragg band gap effect is almost invisible on the FRF curve due to the existence of a prominent transmittance peak inside the band gap, which is addressed in Sec. 3.2.2. Remember that we have demonstrated in Fig. 16(b) that such a transmittance peak inside the Bragg band gap can be simply eliminated by modifying the global resonator location in the finite LR beam. We further verify this idea experimentally. Figure 24 shows the numerical and experimental transmittance FRFs for the modified LR beam specimen, fabricated with $a_L = 105$ mm. It is seen that both the numerical and experimental
between the LR and Bragg band gaps is evidenced in Fig. 20, in frequency is tuned from 777 Hz to 1269 Hz. The near-coupling effect beam-like resonator is tuned by changing the total length (denoted model is presented to facilitate the design and modeling of the design formulas. In particular, an effective lumped-parameter provided a systematic method to characterize band gap behavior of both LR and Bragg band gaps. Furthermore, approximate but examined numerically. In particular, the influence of structural length but highly relevant to the global resonator location (relative position between the resonator array and the finite host beam). The band gap transmittance peak may be eliminated by an appropriate choice of the global resonator location.

The effects of damping on band gap behavior of infinite systems and vibration transmittance of finite structures are studied theoretically. It is shown that the host-beam damping has little influence on the wave attenuation behavior inside band gaps but can exert noticeable influence on the performance within pass bands and at band gap edges. In contrast, the resonator damping can affect the behavior inside and around the LR band gap in a remarkable way.

Following the theoretical investigations, we further conduct experimental measurements of vibration transmittance FRFs for several LR beam specimens designed with gradually tuned local resonances. The experimental results confirm the theoretical predictions of the coexistence and evolution of LR and Bragg band gaps. In particular, three interesting band gap phenomena are evidenced experimentally: (1) transition between LR and Bragg band gaps; (2) near-coupling effect of the local resonance and Bragg scattering; and (3) resonance frequency of local resonators outside of the LR band gap.

The results presented in this paper can facilitate the design and understanding of band gaps in LR beams with beam-like resonators, which can find applications in the control of vibration and wave propagation in flexural beams. The idea can be further extended to other engineering structures, such as flexural plates [34], lattice structures [13–16], etc.

4 Conclusions

In this work, we present a design of LR beam where the local resonators are simply realized by double-ended thin beams with their center attached to a host beam. Such double-ended beams are referred to as beam-like resonators in this paper. We have provided a systematic method to characterize band gap behavior of the proposed LR beam structures, accompanied with explicit design formulas. In particular, an effective lumped-parameter model is presented to facilitate the design and modeling of the beam-like resonators.

The evolution of band gaps in the proposed LR beam structures with tuning-beam-like resonators is studied numerically. The beam-like resonator is tuned by changing the total length (denoted by 2l) but retaining the total mass. We provide a band gap map as a function of the changing 2l, which clearly shows the evolution of the location, the width, and the wave attenuation performance of both LR and Bragg band gaps. Furthermore, approximate but analytical equations are derived to understand the evolution of band gap location and size.

The vibration transmittance FRFs of finite LR beams are also examined numerically. In particular, the influence of structural length and global resonator location is studied. We show that the frequency of band gap transmittance peak (transmittance peak on the FRF within band gap frequency range) is insensitive to the

Fig. 24 Numerical and experimental vibration transmittance FRF for the modified version (a₀ = 105 mm) of the LR beam with resonators tuned at f₀ = 869 Hz

results clearly show up the existence of the Bragg band gap. In Fig. 24, the discrepancy between the numerical and experimental transmittance peaks inside the LR band gap may also be attributed to the simplified treatment of the load, as discussed previously on similar observations in Fig. 21(a). Of particular interest is that the experimental results presented above give the evidence of three interesting band gap phenomena mentioned in Sec. 3.2.1: (1) transition between LR and Bragg band gaps; (2) near-coupling effect of the local resonance and Bragg scattering; and (3) resonance frequency of local resonators outside of the LR band gap. The band gap transition phenomenon is confirmed by the evolution of the experimental transmittance drops observed in Figs. 18–20 and in Fig. 24 as the resonance frequency is tuned from 777 Hz to 1269 Hz. The near-coupling effect between the LR and Bragg band gaps is evidenced in Fig. 20, in which the experimental FRF displays a broadband vibration attenuation induced by such a band gap near-coupling effect. The situation of resonance frequency of local resonators being outside of the LR band gap is evidenced by the case shown in Fig. 22, in which the experimental FRF shows no obvious transmittance drop at the resonance frequency f₀.

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References
