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Analysis and experimental realization of locally resonant phononic plates carrying a periodic array of beam-like resonators

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Abstract
We present theoretical examination and experimental demonstration of locally resonant (LR) phononic plates consisting of a periodic array of beam-like resonators attached to a thin homogeneous plate. Such phononic plates feature unique wave physics due to the coexistence of localized resonance and structural periodicity. We demonstrate that a low-frequency complete band gap for flexural plate waves can be created in the proposed structure owing to the interaction between the localized resonant modes of the beam-like resonators and the flexural wave modes of the host plate. We show that the location and width of the complete band gap can be dramatically tuned by changing the properties of the beam-like resonators. To understand the opening mechanism and evolution behaviour of the complete band gap, some approximate but explicit models are provided and discussed. We further perform experimental measurements of a specimen fabricated by an array of double-stacked aluminum beam-like resonators attached to a thin aluminum plate with 5 cm structure periodicity. The experimental results evidence a complete band gap extending from 465 to 860 Hz, matching well with our theoretical prediction. The LR phononic plates proposed in this work can find potential applications in attenuation of low-frequency mechanical vibrations and insulation of low-frequency audible sound.

Keywords: phononic plate, local resonator, complete band gap, phononic crystal, flexural wave

(Some figures may appear in colour only in the online journal)

1. Introduction

The past few years have witnessed a growing interest for the propagation of elastic waves in periodic structures known as phononic crystals and phononic metamaterials [1–16]. The focus has been placed on the existence of frequency regions, known as complete band gaps, in which no elastic wave can propagate in the periodic structures, whatever the direction of propagation. Such a unique property can find applications in the design of sound barriers, vibration isolators, acoustic transducers, mechanical waveguides, etc.

Existing investigations on phononic crystals/metamaterials have revealed two mechanisms responsible for the creation of band gaps. One is known as the Bragg scattering mechanism, resulting from the spatial periodicity of the structure. Another is known as locally resonant (LR) mechanism, originating from the resonant modes of microstructures in each unit cell. The LR mechanism was first proposed by Liu et al [3]. They fabricated an LR phononic crystal by embedding an array of resonant microstructures into a matrix material, and provided theoretical and experimental evidence for the opening of LR band gaps in a frequency range two orders of magnitude lower than that given by the Bragg scattering mechanism. Subsequent studies on several types of LR phononic crystals demonstrated that
such periodic structures may exhibit negative mass density [4], negative modulus [7] and double negative properties [6]. Thus they may be designated as phononic metamaterials, as an analogue with photonic metamaterials.

The concept of LR phononic crystals/metamaterials has been implemented in the design of LR phononic plate structures [17–32]. One design strategy is based on introducing resonant inclusions into a structured periodic plate. For example, Hsu and Wu proposed a design of LR phononic plate by embedding soft rubber inclusions periodically into a thin elastic plate [20] and showed the existence of narrow complete LR band gaps. Yu et al. [32] suggested a design based on placing coated blocks in holes drilled periodically in a host thin plate, in which the block is high density solid and the coating is an elastically soft material. They experimentally evidenced the existence of a very low-frequency band gap.

Another design strategy is based on depositing resonant microstructures on a homogeneous plate. For instance, Wu et al. [17, 18] and Pennec et al. [23, 24] reported independently the design and analysis of LR phononic stubbed plates consisting of a periodic array of cylindrical stubs deposited on a thin plate of homogeneous material. They showed that a complete LR band gap can be created in the structure due to the coupling between the resonant modes of the stubs and the plate wave modes. Recently, Oudich et al. presented experimental evidence of the existence of a low-frequency LR band gap in similar LR stubbed plates [25]. They measured a complete band gap exhibiting in an audible frequency range for the flexural wave modes. To achieve more design feasibilities, Oudich et al. [26] and Assouar et al. [27, 28] suggested the use of two-layer composite stubs to construct LR phononic stubbed plates. A similar idea was examined by Hsu et al. [21, 22], who considered stubs consisting of two cylinder segments with different radii. More recently, Xiao et al. [29] proposed an LR phononic plate carrying a square array of simple spring–mass resonators. Such a structure represents an elegant physical model for the understanding of basic wave physics in LR phononic plates. Torrent et al. [31] subsequently examined a similar LR plate system carrying a honeycomb array of spring–mass resonators. They demonstrated that such an LR plate structure can be considered as an elastic analogue of graphene, exhibiting very interesting wave phenomena. Xiao et al. [30] also investigated the sound insulation performance of LR phononic plates with multiple arrays of local resonators. They found that such plate structures can result in much higher low-frequency sound transmission loss than a bare plate with the same surface mass density. The existing study of LR phononic plates promised their potential applications in high frequencies for filtering and sensing, as well as in low frequencies for vibration attenuation and sound insulation.

In the present work, we propose a new design of LR phononic plate, constructed by attaching a two-dimensional (2D) periodic array of beam-like structures to a thin homogeneous plate. The beam-like structures are referred to as beam-like resonators in this paper due to their exhibition of low-frequency resonant modes. The proposed new design of LR phononic plate actually represents an extension of our previously proposed 1D LR phononic beam [33] to its 2D counterpart. In our previous work [33], we have evidenced that a 1D LR phononic beam with beam-like resonators can exhibit LR and Bragg band gaps simultaneously, and the properties of these band gaps can be remarkably influenced by changing the length of beam-like resonators. It is expected that the 2D LR phononic plate considered in this work may exhibit some basically similar behaviour as the 1D case of LR phononic beam. However, more attention still needs to be paid to the 2D case for the following reasons. First, the band-gap behaviour of the 2D case is more complex than the 1D case. In particular, the band-gap behaviour of a 2D LR phononic plate is direction dependent due to the anisotropic effects induced by the 2D arrangement of local resonators. Second, different theoretical methods and experimental tools are required for the 2D case to achieve a more feasible and comprehensive characterization of the band-gap behaviour. Third, in view of the rich wave physics and various potential applications demonstrated by the growing publications on LR phononic plates [17–32], the design, examination and realization of an alternative construction of 2D LR phononic plates, as schemed in this work, shall add new knowledge to this area, and are expected to be helpful for possibly further explorations on this sort of 2D LR phononic plates.

In general, the purpose of this work is to present a detailed theoretical examination of the flexural wave band-gap behaviour of the proposed 2D LR phononic plate structure, as well as to provide experimental evidence for the existence of complete band gap. To predict band-gap behaviour, we adopt a $k(\omega)$-form plane wave expansion (PWE) method, which has been developed in [29] to deal with a simple LR phononic plate carrying spring–mass resonators. The $k(\omega)$-form PWE method enables the calculation of imaginary parts of Bloch wavenumbers Im($k$) for a given frequency $\omega$. Thus the wave attenuation performance inside a band gap can be well quantified. In our theoretical examinations, we show how the band-gap behaviour can be dramatically tuned by changing the thickness, the width and the length of the beam-like resonators. Besides the theoretical analysis, we experimentally demonstrate that the proposed structure can be easily fabricated using a single metallic material, and show that a fabricated sample can realize a complete band gap at very low audible frequencies (<1000 Hz) with a reasonable lattice size (5 cm). Our experimental measurements are performed by using a scanning laser Doppler vibrometer which can easily record the out-of-plane response of the entire host plate. The theoretical and experimental results of this work suggest that the proposed LR phononic plates can find applications in reduction of low-frequency mechanical vibrations and insulation of low-frequency audible sound.

The paper is organized as follows. Section 2 presents the configuration of the proposed LR phononic plate structure as well as the theoretical method for the prediction of band-gap behaviour. Section 3 describes the LR phononic plate specimen fabricated in this work, as well as the experimental setup and measurement method. In section 4, we first perform an in-depth theoretical examination of the band-gap behaviour, and then present experimental characterization of the band-gap effects in our fabricated specimen. The measured
complete band gap shows very good agreement with theoretical predictions. Finally, this work is concluded in section 5.

2. Proposed design and modelling method

The proposed design of LR phononic plate is shown in figure 1. The structure consists of a periodic array of beam-like resonators with their centre attached to a thin homogeneous plate. The mounting of the beam-like resonators is realized by a short, thin rectangular washer. For the example depicted in figure 1, the attached beam-like resonators make an angle of about 45° with respect to the axes of the lattice. Such an orientation can make the length of the beam-like resonators large. Actually, the choice of the orientation for beam-like resonators has little influence on the results. This point has been verified by our finite element simulations on a group of examples, but the results are not presented here for the sake of brevity. Moreover, it should be mentioned that at each mounting location multiple beam-like resonators may be stacked through multiple washers. In figure 1, we only depict the case of double-stacked identical beam-like resonators. The dimension of the beam-like resonators and washers are 2l × w × t and lw × w × t, respectively.

In this work, all structural components of the proposed LR phononic plate shown in figure 1 are considered to be made of metallic material. Since our examination will be restricted to a low-frequency range where the structural wavelength is considerably larger than the dimension of the washer, our modelling can be simplified by assuming that each washer act as a rigid point. In addition, in this work we are only interested in the low-frequency flexural wave band gaps of the LR phononic plate. We may expect that several types of beam vibration modes (of the beam-like resonators) can interact with the flexural wave motion of the host plate to generate flexural wave band gaps. Such beam vibration modes include the torsional modes, the antisymmetric flexural modes (with the two cantilevers of the attached beam vibrating out of phase), and the symmetric flexural modes (with the two cantilevers vibrating in phase). The torsional modes and antisymmetric flexural modes of the attached beam both exert a twisting moment to the host plate. Our previous work on the case of LR phononic beams [33] with periodic beam-like resonators has demonstrated that such twisting moment exerted by the attached thin beams is generally too weak to affect the wave characteristics of the host flexural structure. In contrast, the symmetric flexural modes of the attached beams exert a transverse force to the host structure and it does strongly interact with the flexural wave motion of the host structure to create flexural wave band gaps. Therefore, in the theoretical modelling of this work we will neglect the torsional modes and antisymmetric flexural modes of the attached beam-like resonators, but account the symmetric flexural modes. The validity of these assumptions will be confirmed by numerical simulations and experimental measurements.

The dynamic effects produced by the symmetric flexural modes of an attached beam-like resonator can be characterized by its driving-point dynamic stiffness Dr (the ratio of the driving force to the resultant displacement at the driving point). Following the Bernoulli–Euler bending theory, the driving-point dynamic stiffness Dr can be derived to be [33]

\[
D_r = -\omega^2 \rho_c A_t l_w + \frac{-2E_t I_t (\beta l_r)^3 \sin(\beta l_r) \cosh(\beta l_r) + \cos(\beta l_r) \sinh(\beta l_r)}{I^3} \frac{1}{1 + \cos(\beta l_r) \cosh(\beta l_r)},
\]

where \( \omega \) is the circular frequency, \( l_r = (2l - l_w)/2 \) refers to the length of the cantilever beam segment, \( \rho_c \) and \( E_t \) are the mass density and Young’s modulus of the beam material, \( A_t \) and \( I_t \) are the cross-sectional area \( (w \times t) \) and second moment of area \( (w \times t^3/12) \) of the beam. In addition, \( \beta \) is the flexural wavenumber of the beam, given by

\[
\beta = \left( \frac{\rho_c A_t \omega^2}{E_t I_t} \right)^{1/4}.
\]

It should be noted that if \( N \) beam-like resonators are stacked at the same position as shown in figure 1 for the double-stacked case \( (N = 2) \), the total driving-point dynamic stiffness of the attached beam-like resonators, \( D_R \), is given by the sum of each individuals, i.e.

\[
D_R = \sum_{j=1}^{N} D_{r,j},
\]

where \( D_{r,j} \) is the driving-point dynamic stiffness of the \( j \)th beam-like resonator. If the individual beam-like resonators are all identical, we can simply obtain \( D_R = ND_r \).

In figure 1, the attachment location of a beam-like resonator can be represented by the following lattice vector

\[
\mathbf{R} = \bar{m} a_1 + \bar{n} a_2.
\]

where \( \bar{m} \) and \( \bar{n} \) are integers, \( a_1 = (a_{1x}, a_{1y}) \) and \( a_2 = (a_{2x}, a_{2y}) \) are basis vectors of the direct lattice. In this work, we only consider the case of square lattice, with the basis vectors defined by \( a_1 = (a, 0) \) and \( a_2 = (0, a) \).
Following the classical thin plate theory, the governing equation for the time-harmonic flexural vibration of the host plate is given by [29]

\[
DV^4W(r) - \omega^2 \rho h W(r) = \sum_R [F(R)\delta(r - R)],
\]

where \(V^4 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2\), \(D = Eh^3/12(1 - \nu^2)\) is the plate bending stiffness, \(\rho\) is the mass density of the plate material, \(h\) is the thickness of the plate, \(r = (x, y)\) denotes the location, \(W(r)\) refers to the transverse displacement of the plate, and \(F(R)\) represents the force applied to the plate by the beam-like resonator attached at \(R\), which are related to the plate displacement such that

\[
F(R) = -D_R W(R),
\]

where \(D_R\) is given by equation (3).

Substituting equation (6) into equation (5) gives

\[
(DV^4 - \omega^2 \rho h)W(r) = \sum_R [-D_R W(R)\delta(r - R)].
\]

Due to the periodicity of the system the plate displacement can be expanded as

\[
W(r) = \sum_G W_G e^{-i(kG \cdot r)},
\]

where \(k = (k_x, k_y)\) is the Bloch wave vector and \(G\) is the reciprocal lattice vector, given by

\[
G = mb_1 + nb_2,
\]

with \(m\) and \(n\) being integers, \(b_1 = (b_{1x}, b_{1y})\) and \(b_2 = (b_{2x}, b_{2y})\) being the basis vectors of the reciprocal lattice. For the case of square lattice considered in this work, we have \(b_1 = (2\pi/a, 0)\) and \(b_2 = (0, 2\pi/a)\).

Inserting the expanded form of \(W(r)\) into equation (7) gives

\[
\sum_G (D|k + G|^4 - \rho h \omega^2) W_G e^{-i(kG \cdot r)} = \sum_G \sum_R [-D_R W_G e^{-i(kG \cdot r)}].
\]

with \(G^*\) and \(G^*\) both being the reciprocal lattice vector.

Multiplying equation (10) by \(\exp[i(k + G) \cdot r]\) and integrating it over one unit cell yields

\[
(DS|k + G|^4 - \rho h S \omega^2) W_G + \sum_G D_R W_G = 0.
\]

where \(S\) is the area of the unit cell. To solve equation (11) we should truncate the summations in equation (8) by choosing \(m, n = (-M, \ldots, M)\). Then the number of plane waves \((W_G)\) used in the calculation is \(N^2 = (2M + 1)^2\). Therefore equation (11) can be written as the following matrix form

\[
(DS[K] - \rho h S \omega^2[I] + D_R[U])\{W\} = 0.
\]

where

\[
[K] = \\
\begin{bmatrix}
|k + G|^4 & 0 & \ldots & 0 \\
0 & |k + G|^4 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & |k + G|^4 \\
\end{bmatrix}
\]

and

\[
[I] = \\
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 1 \\
\end{bmatrix}_{N^2 \times N^2}.
\]

Notice that the Bloch wave vector \(k\) can be written as

\[
k = (k_x, k_y) = (k \cos \phi, k \sin \phi),
\]

where \(k\) refers to the Bloch wavenumber, which can be a complex value, whose real part denotes the phase change per unit length corresponding to a propagating Bloch wave, and imaginary part quantifies the exponential decay rate of an evanescent Bloch wave. In addition, \(\phi\) denotes the azimuth angle (wave direction) associated with the Bloch wave vector \(k\).

Thus, for the case of square lattice to be considered in this work, the term \(|k + G|^4\) in equation (11) can be expressed by

\[
|k + G|^4 = \frac{1}{d^4}[(ka)^4 + \gamma_1(ka)^3 + \gamma_2(ka)^2 + \gamma_3(ka) + \gamma_0],
\]

where

\[
\gamma_0 = 4\pi^2(m^2 + n^2), \quad \gamma_1 = 4\pi(m \cos \phi + n \sin \phi),
\]

\[
\gamma_2 = 2\pi \epsilon_0, \quad \gamma_3 = 2\pi \epsilon_1.
\]

Therefore the matrix \([K]\) in equation (12) can be written as

\[
[K] = \frac{1}{d^4}[(ka)^4[I] + (ka)^3[K_3]] + (ka)^2[K_2] + (ka)[K_1] + [K_0],
\]

where

\[
[K_{j=0,1,2,3}] = \\
\begin{bmatrix}
\gamma_{j,1} & 0 & \ldots & 0 \\
0 & \gamma_{j,2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \gamma_{j,N^2} \\
\end{bmatrix}_{N^2 \times N^2}.
\]

Substituting equation (17) into (12) gives

\[
[\tilde{k}^4 I + \tilde{k}^3 A_3 + \tilde{k}^2 A_2 + \tilde{k} A_1 + A_0][W] = 0
\]

where

\[
A_{1,2,3} = [K_{1,2,3}], \quad A_0 = [K_0] - [\tilde{k}^4 I] + D_R[U],
\]

\[
\tilde{k} = ka, \quad \tilde{k}^4 = \frac{\rho h \omega^2 a^4}{D}, \quad D_R = \frac{D_R a^4}{DS}.
\]
Equation (19) represents a quartic eigenvalue problem for $\bar{k}$. Such a problem can be further recast as the standard linear eigenvalue problem as described in [29]. Using this formulation, for each given frequency $\omega$ we can calculate a series of solutions of Bloch wavenumber $k$, with generally complex values, whose imaginary part $\text{Im}(k)$ quantifies wave attenuation performance. Such a method of predicting dispersion relations has been known as a $k(\omega)$-form PWE method in the literature [34, 35]. It should be mentioned that, although a series of solutions of $k$ exist, the lowest component whose real part $\text{Re}(k)$ lies inside and around the first Brillouin zone is the most accurate one. Thus in this work we will present only the predictions associated with the lowest component.

3. Specimen and experimental setup

We fabricate the specimen of LR phononic plate using a 1 mm thick aluminum plate attached to a square array of $10 \times 10$ double-stacked identical beam-like resonators, as shown in figure 2(a). The specimen of the beam-like resonators are also made of aluminum material (Young’s modulus $E = 7 \times 10^{10}$ Pa, density $\rho = 2700$ kg m$^{-3}$, Poisson’s ratio $\nu = 0.3$), being the same as the host thin plate. The lattice constant of the resonator array is $a = 0.05$ m. The geometric parameters of the beams and washers are: $2l = 0.1$ m, $w = 0.014$ m, $t = 0.002$ m, $l_w = 0.004$ m. The dimension of the host plate is $0.84$ m $\times$ $0.84$ m.

In our experiment, the LR phononic plate specimen is hung up through two soft springs to simulate free boundary conditions. The specimen is excited at its centre point by the B&K vibration exciter 4824, which is fed with a magnified pseudo random signal up to 3.2 kHz. In figure 2(b), the vibration exciter is placed behind the specimen, and it is connected to the back plate surface over which the beam-like resonators are attached. The out-of-plane response is measured by a scanning laser Doppler vibrometer (SLDV, Polytec PSV400M2), with its laser beam scanned over the front plate surface observed in figure 2(b). Four measurements are conducted following different scan strategies. The first three measurements are performed by scanning 60 points along three different directions ($0^\circ$, $45^\circ$, $90^\circ$) respectively (figure 2(c)). The last measurement is performed by scanning $60 \times 60$ points over $0.45$ m $\times$ $0.45$ m area corresponding to the location of the resonator array (figure 2(c)).

The displacement frequency response function (FRF) of each scan point $u_j(f)$ is obtained by the SLDV through a fast Fourier transform (FFT) of the recorded time traces $u_j(t)$, $j = 1, \ldots, n$, where $n$ represents the total number of scan points, $t$ is time and $f$ is frequency. In each measurement, the software also calculates an averaged frequency response spectrum across all scan points according to the following averaging procedure

$$\bar{u}(f) = \frac{1}{n} \sum_{j=1}^{n} |u_j(f)|.$$ (21)

The plot of the averaged frequency spectrum $\bar{u}(f)$ as a function of frequency $f$ can be utilized to demonstrate the existence of band gaps within which a significant drop of $\bar{u}$ can be observed. This method of band-gap characterization has been employed by Assouar et al [25, 28]. An alternative characterization method is to calculate/measure the vibration transmittance FRF [29, 33] across a finite extent of the structure. For the finite structure shown in figure 2(c), if we denote the displacement FRF of the excitation point as $u_A$, and that of the left-bottom corner (point B) of the scanned area as $u_B$, the vibration transmittance FRF across the two points is given by [29, 33]

$$T = \left| \frac{u_B}{u_A} \right|.$$ (22)

4. Results and discussion

4.1. Theoretical examination

Before presenting the experimental results, we first perform a theoretical study on the band-gap behaviour of the proposed LR phononic plate. Figure 3(a) shows the evolution of band-gap behaviour with the wave direction $\phi$ (see equation (14)). In this figure, the white areas represent pass bands, within which there...
exists at least one propagating mode (Im(k) = 0), while the other coloured areas characterize band gaps where all wave modes are evanescent (Im(k) > 0). It should be pointed out that, at a band-gap frequency, generally there exist many components of evanescent wave modes and each component is characterized by a solution of Im(k) > 0, but the overall wave attenuation performance is mostly quantified by the magnitude of the smallest Im(k), which represents the least rapidly decaying wave that carries energy the farthest. For this reason, in the band-gap regions shown in figure 3(a), we only present the solution of the smallest Im(k), whose magnitude is defined by the colour bar on the right side of the plot. Thus, the colour bar in figure 3(a) just describes the band-gap regions where Im(k) > 0, but have no reference to the pass-band regions where Im(k) = 0. We adopt such a plot strategy for presenting a clearer illustration of the band-gap regions with observable bounding edges. As can be seen, the coloured band-gap map presented in figure 3(a) simultaneously displays the location, width and wave attenuation performance of band gaps along any propagation direction within the angle ranging from 0 to 45°. In fact, the performance along any other direction outside this angle range can be immediately obtained by accounting the symmetry of the square lattice. Generally speaking, the Im(k)-form band-gap characterization method represented by figure 3(a) can be considered as an alternative method compared to the conventional Re(k)-form band structure descriptions which only display the location and width of band gaps. Actually, a traditional Re(k)-form band structure associated with this example is presented later in figure 6 (solid-curve case in the central panel).

It is seen in figure 3(a) that there exist two band gaps for a given wave direction φ. The location of the upper band gap gradually increases as the angle φ changes from 0 to 45°. In particular, one edge of this band gap is governed by the Bragg condition associated with the periodic square lattice, and the associated edge frequencies actually represent the Bragg frequencies, given by

\[ f_B(φ) = \frac{1}{2\pi} \left( \frac{π}{a \cdot \cos φ} \right)^2 \sqrt{\frac{D}{\rho h},} \]  

which depends on the lattice constant a, but has no reference to the properties of beam-like resonators. Accordingly, the upper band gap observed in figure 3(a) should be regarded as a Bragg band gap. Note that equation (23) is derived from the Bragg conditions along different directions, i.e. \( a \cdot \cos φ = \frac{λ_p}{2} \), where \( λ_p \) refers to the flexural wavelength of the plate, which is given by \( λ_p = \frac{2\pi / k_p}{ρ} \), where \( k_p = \sqrt{\frac{E}{\rho h}/D} \) represents the flexural wavenumber of the plate.

In contrast to the upper band gap, the location of the lower band gap in figure 3(a) remains almost unchanged. This can be understood by the lowest symmetric flexural resonance frequency of the beam-like resonators, given by [33]

\[ f_l = \frac{1}{2\pi} \sqrt{\frac{1.8754}{l^2} E t^2}, \]  

which is derived according to the condition \( 1 + \cos(β_l r) \cos(β_l r) = 0 \), under which the driving-point dynamic stiffness \( D_l \) (see equation (1)) of the resonator is infinitely large. By using equation (24) we can immediately obtain \( f_l = 713.9 \text{ Hz} \). It can be seen the resonance frequency \( f_l \) indeed lies inside the lower band gap. Thus the lower band gap should be regarded as an LR band gap, whose location is highly relevant to the properties of local resonators. From the map of the lower band gap we can further identify a complete band gap within which the condition Im(k) > 0 is satisfied along all directions. The identified complete band gap extends from 484 Hz to 883 Hz, as denoted by the two horizontal dashed–dotted lines in figure 3(a). To verify the existence of the complete band gap, we further calculate the vibration transmittance FRF across the points A and B of the finite structure illustrated in figure 2(c). The calculation is carried out by using the finite element method (FEM), performed by the well-known FE package: MSC. Nastran. The results are shown in figure 3(b). It is seen the transmittance FRF exhibits a noticeable drop in a frequency range which is well consistent with the complete band gap (shaded region) predicted in figure 3(a).

Although a continuous beam-like resonator can produce multiple flexural resonances, the results of figure 3(a) implies
that only the lowest resonance (at 713.9 Hz) is responsible for the formation of the LR band gap. This is as expected, since in the low-frequency range considered here, the attached continuous beam-like resonator can be approximated by a lumped mass–spring–mass system, where the former lumped mass is assumed to be rigidly attached to the host plate, and the rest spring–mass system acts as a single-degree-of-freedom resonator. If we denote the properties of the three components of the approximate lumped system by the parameters $m_0$, $k_1$ and $m_r$, they can be determined by the following equations [33]:

\[ m_0 = \rho A_1 l_w + 0.395 \rho A_r (2l_r), \]
\[ m_r = 0.605 \rho A_r (2l_r), \]
\[ k_1 = 14.953 E_r l_r / l^3, \]

where $l_r = (2l - l_w)/2$. The three expressions of equation (25) are derived according to such prescribed conditions [33]; (i) the approximate lumped system has the same total mass as the original continuous system; (ii) the resonance and anti-resonance frequencies of the approximate system are respectively equal to the lowest resonance and anti-resonance frequencies of the original continuous system. Note that the lowest resonance and anti-resonance frequencies of the original system can be directly obtained from equation (1), which implies that these two characteristic frequencies are determined by the conditions $1 + \cos \beta l_i \cosh \beta l_r = 0$ and $\sin \beta l_i \cosh \beta l_r + \cos \beta l_i \sinh \beta l_r = 0$, respectively.

The driving-point dynamic stiffness of the approximate lumped system is given by

\[ \bar{D}_1 = -\omega^2 m_0 + \frac{-\omega^2 k_1 m_r}{k_r - \omega^2 m_r}, \]

which actually represents a low-frequency approximation of the expression given by equation (1). Note that for the case of double-stacked identical beam-like resonators, the total driving-point dynamic stiffness of its approximate lumped system is given by $D_R = 2D_1$.

To demonstrate the validity of the approximation, figure 4 shows the comparison of the driving-point dynamic stiffness and the directional band-gap behaviour calculated by the beam model and the approximate mass–spring–mass model. Figure 4(a) shows a very good agreement between these two models, especially in the low-frequency range. In particular, at frequencies below 1200 Hz, which are interested in this work, the agreement is perfect. This is further confirmed by the comparison of band-gap behaviour presented in figure 4(b).

To understand the evolution of directional band-gap region in figure 3(a), we can utilize a 1D approximation of the 2D periodic system along different directions. For a given wave direction $\phi$, the 2D LR plate can be approximated by a 1D periodic system consisting of the host plate attached with a periodic array of line beam-like resonators, lying perpendicular to the wave direction. In the approximate model, the interval between adjacent line resonators is $a \cos \phi$, and the driving-point dynamic stiffness of the line resonators is $D_R \cos \phi / a$ per unit length. Such an approximate system with 1D periodicity can be modelled analytically by using a spectral element method [33, 36], and then an explicit dispersion relation can be obtained as

\[ \cosh^2(ik \bar{a}) + \alpha_1 \cosh(ik \bar{a}) + \alpha_2 = 0, \]

where

\[ \alpha_1 = -[\cos(kp \bar{a}) + \cosh(kp \bar{a})] \]
\[ -\left(\frac{P}{4}\right)\sin(kp \bar{a}) - \sinh(kp \bar{a}) \],
\[ \alpha_2 = \cos(kp \bar{a}) \cosh(kp \bar{a}) + \left(\frac{P}{4}\right)\sin(kp \bar{a}) \cosh(kp \bar{a}) \]
\[ -\sin(kp \bar{a}) \cosh(kp \bar{a}) \],
\[ \bar{a} = a \cos \phi, \quad P = D_R \cos \phi / D a k_p^3 \].

where $k_p = (\rho \omega^2 / D)^{1/4}$ denotes the flexural wavenumber of the plate.

It should be mentioned that the basic idea of the above approximation method has been described in [29], where only the approximation along the $\Gamma X$ direction was treated, and the attached local resonators addressed there were restricted to spring–mass systems.

We notice that the band edges (intersections of the pass bands and the boundaries of the irreducible Brillouin zone, i.e. $k \bar{a} = \pi$ and $k \bar{a} = 0$) of the approximate 1D periodic system are governed by

\[ \cosh(ik \bar{a}) = \pm 1. \]
Inserting equation (29) into (27) yields the following four types of band-edge frequencies:

\[ r_B : \cos(kp_0 \cos \phi) = -1, \]  
\[ r_R : \frac{D_R}{4Da_k^2} \cos \phi = \left( \tan \frac{k_p a \cos \phi}{2} - \tan \frac{k_p a \cos \phi}{2} \right)^{-1}, \]  
\[ s_B : \cos(kp_0 \cos \phi) = 1, \]  
\[ s_R : \frac{D_R}{4Da_k^2} \cos \phi = \left( \coth \frac{k_p a \cos \phi}{2} + \cot \frac{k_p a \cos \phi}{2} \right)^{-1}. \]

The \( r_B \) and \( s_B \) frequencies given by equations (30) and (32) actually represent the odd and even order Bragg frequencies of the 1D periodic lattice, i.e.

\[ r_B : f_{B,n}(\phi) = \frac{1}{2\pi} \left( \frac{n\pi}{a \cdot \cos \phi} \right)^2 \sqrt{\frac{D}{\rho h}}, n = (1, 3, 5, \ldots), \]
\[ s_B : f_{B,n}(\phi) = \frac{1}{2\pi} \left( \frac{n\pi}{a \cdot \cos \phi} \right)^2 \sqrt{\frac{D}{\rho h}}, n = (2, 4, 6, \ldots). \]

Figure 5 shows the evolution of band-edge frequencies predicted based on the 1D approximate model of the system considered in figure 3.

Figure 6. Effects of the number of stacked beam-like resonators (\( N \)) on the band-gap behaviour.
beam-like resonators have demonstrated that, if the resonance frequency of beam-like resonators is gradually increased, the lowest band gap of the LR beam will experience a transition from an LR band gap to a Bragg band gap [33]. That is, before the transition state, the lowest gap is an LR band gap, whose location gradually increases with increasing resonance frequency; while after the transition state, the lowest gap becomes a Bragg band gap, whose location remains unchanged around the lowest Bragg frequency. Figure 8(a) shows a very similar phenomenon as the LR beam. The transition occurs in figure 8(a) due to that the resonance frequency \( f_r \) (equation (24)) increases with increasing beam thickness \( t \). When \( t > t_0 \), the lowest \( \Gamma X \) directional gap is of the LR-type, and when \( t < t_0 \), this directional gap changes to the Bragg-type.

An additional interesting feature observed in figures 7(a) and 8(a) is that the transition condition \( t = t_0 = 2.22 \times 10^{-3} \text{ m} \) can be approximately predicted by the intersection point of the \( r_B \) (equation (34)) and \( s_R \) (equation (33)) band-edge frequency curves (\( \Gamma X \) direction), which gives an approximate transition condition: \( t = 2.19 \times 10^{-3} \text{ m} \), being very close to the true condition \( t = t_0 = 2.22 \times 10^{-3} \text{ m} \).

Figure 7(b) shows that changing the beam width \( w \) can affect the width of the complete band gap, but has little influence on the gap location. This is because the gap location is dominated by the resonance frequency \( f_r \), which is given by equation (24) and has no reference to the parameter \( w \). It is seen in figure 7(b) that the gap width increases with increasing \( w \). The reason is that a larger \( w \) results in a heavier resonator mass \( m_r \) (equation (25)), and thus gives rise to a stronger resonator effect to form a wider LR gap.

Figures 7(c) and (d) show the evolution of the complete band gap with varying beam length \( 2l \) under two different tuning strategies. In figure 7(c) only the parameter \( 2l \) is changed so that a larger \( 2l \) will result in a heavier beam-like resonator, while in figure 7(d) the parameters \( 2l \) and the \( w \) are tuned simultaneously to keep the resonator mass as a constant. It is seen that in both cases the location and width of the complete band gap can be significantly tuned with varying
The second discrepancy between figures 9 should be somewhat lower than the theoretical counterparts. The resultant band-gap location in the experimental sample assumption. Therefore, the local resonance frequency and beam-like resonators are realized by a layer of glue, which host plate, but in our experimental sample the mountings of beams with local resonators have demonstrated that the amount of vibration attenuation inside the LR band gap should be smaller than the theoretical counterpart considered here, if we take account of the resonator damping, which is excluded in the numerical model. Our previous study on the case of 1D LR phononic beams with local resonators has demonstrated that the amount of maximum vibration attenuation inside an LR band gap decrease significantly with increasing resonator damping. Thus it would be expected that for the case of LR phononic plate considered here, if we take account of the resonator damping, the actual amount of maximum vibration attenuation inside the LR band gap should be smaller than the theoretical counterpart predicted based on the undamped case.

Figure 10 shows the measured average frequency response spectrum $\tilde{u}(f)$ following different scan strategies as illustrated in figure 2(c): (a) along direction $0^\circ$; (b) along direction $45^\circ$; (c) along direction $90^\circ$; (d) over the square area corresponding to the location of resonator array. The shaded frequency region corresponds to the complete band-gap range predicted in figure 3(a). (figure 9(a)) is generally much smaller than that predicted by the theoretical result (figure 9(b)). One main reason is that, at the frequencies with significant vibration attenuation, the vibration response at point B is too small to be accurately recorded by the SLDV. As a result, the vibration transmittance drop is underestimated. Another possible reason may be due to the structural damping which is excluded in the numerical model. In this section, we present the experimental characterization of the fabricated LR plate sample described in section 3. The main purpose is to provide experimental evidence for the existence of the complete band gap that has been numerically predicted in figure 3.

Figure 9 shows the comparison between the experimental and theoretical vibration transmittance FRFs across points A and B. Generally, good agreement can be seen in respect to the frequency range exhibiting noticeable transmittance drop due to the band-gap effect. However, some discrepancies do exist. First, it seems that the band-gap range of the experimental sample (465–860 Hz) is a little bit lower than the theoretical prediction (484–883 Hz), hence no obvious transmittance drop is observed around the theoretical upper gap edge frequencies. The lowering of the band-gap range is as expected, since in our theoretical model the beam-like resonators are assumed to be rigidly bonded to the washers and rigidly attached to the host plate, but in our experimental sample the mountings of beam-like resonators are realized by a layer of glue, which makes the mounting somewhat softer than the theoretical assumption. Therefore, the local resonance frequency and the resultant band-gap location in the experimental sample should be somewhat lower than the theoretical counterparts. The second discrepancy between figures 9(a) and (b) is found on the amount of transmittance drop. It is shown that the transmittance drop indicated by the experimental result (figure 9(a)) is generally much smaller than that predicted by the theoretical result (figure 9(b)). One main reason is that, at the frequencies with significant vibration attenuation, the vibration response at point B is too small to be accurately recorded by the SLDV. As a result, the vibration transmittance drop is underestimated. Another possible reason may be due to the structural damping which is excluded in the numerical model. In this section, we present the experimental characterization of the fabricated LR plate sample described in section 3. The main purpose is to provide experimental evidence for the existence of the complete band gap that has been numerically predicted in figure 3.

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This is because the distance between the scan points and the excitation point A for the case of direction 0° (figure 2(c)) is averagely larger than the other cases.

The behaviour of the fabricated LR plate sample is further characterized by the measured steady-state vibration profiles at an excitation frequency below the complete gap (figure 11(a)), inside the complete gap (figure 11(b)), and above the complete gap (figure 11(c)). It is interesting to note that, for the case of frequency inside the band gap (figure 11(b)), vibration response is confined into a small area around the excitation point, and outside this area the plate appears to be almost undeformed. In contrast, for the cases of frequencies below (figure 11(a)) and above (figure 11(c)) the band gap, the plate vibrates globally over the entire structure. The results of figure 11 provide additional evidence of the band-gap effects.

5. Conclusions

In this paper, we present a design of LR phononic plate consisting of 2D periodic array of beam-like resonators attached to a thin homogeneous plate. Based on a $k(\omega)$-form PWE method, we derive an explicit matrix formulation for the calculation of complex band structures of flexural waves in the proposed LR phononic plate. The associated band-gap behaviour is theoretically characterized by computing and plotting the imaginary parts of Bloch wavenumber, $\text{Im}(k)$, as a function of wave angle $\phi$ and frequency $f$, which displays the location, width and wave attenuation performance of band gaps along any direction.

We have conducted a detailed theoretical examination of the influence of the properties of attached beam-like resonators on the behaviour of complete band gap in the proposed structure. We show that the complete band gap can be significantly enlarged by increasing the number of stacked beam-like resonators at each attachment point or by increasing the width of the beam-like resonators. We also demonstrate that both the location and width of the complete band gap can be remarkably tuned by changing the thickness and length of beam-like resonators. Of particular interest is the observation of some band-gap transition states in the process of tuning resonator parameters. Such transition states originate from the transition between local resonance and Bragg scattering associated with the $TX$ directional band gap. Besides the theoretical band-gap predictions, we further provide some approximate but explicit models to enable a more physical and feasible understanding of the opening and evolution of band gaps. Finally, we conduct experimental measurements of an LR phononic plate specimen fabricated by a 2D periodic array of double-stacked aluminum beam-like resonators attached to a thin aluminum plate with 5 cm structure periodicity. The experimental results evidence the existence of a complete band gap extending from 465 to 860 Hz, matching well with our theoretical prediction.

The LR phononic plates proposed in this work can find applications in attenuation of low-frequency mechanical waves, and may also be implemented as acoustic barriers for audible sound insulation. In addition, notice our proposed structures can be fabricated using a single metallic material, thus they may be conveniently manufactured, installed and used in harsh environments (e.g., high temperature).

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References
